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Criticality and transient chaos in a sandpile model

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We numerically investigate a coupled map lattice model which is a generalization of the critical height sandpile automaton. In the case of periodic boundary conditions we find in dependence on a threshold parameter strong evidence for a second order phase transition between states of different spatial order. In the disordered phase the spatial structure is irregular with long range linearly decaying correlations. In the ordered phase dynamics is dominated by a few coexisting periodic attractors whose basins of attraction become infinitely small at the critical point. At this point transient lengths diverge and the transients are chaotic. With open boundary conditions the system exhibits self-organized criticality, i.e., adjusts itself to the vicinity of this critical point.

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The formation of scale invariant spatial and temporal structures, i.e., spatial fractals and $1/f$ type noise, is one of the characteristic features of spatially extended nonlinear dynamical systems. A general explanation for these common phenomena is not known yet. But at least some of these phenomena may be explained by the concept of self-organized criticality (SOC) [1]. Systems showing SOC organize themselves into a stationary state which is characterized by scale invariant spatial and temporal correlations. Typically, their time series have an intermittencylike form with bursts having power law distributions of sizes. Examples of self-organized critical phenomena are the dynamics of avalanches on sandpiles under appropriate conditions [2] and on sandpile models [1,3,4], the domain wall growth in magnets [5], earthquakes [6,7], the fragmenting of solid matter [8], and pinned charge-density waves [9].

Our present understanding of SOC is drawn largely from numerical simulations on discrete space-time lattice models, especially cellular automata modeling sandpiles. The main ingredients of these so-called sandpile automata are threshold dynamics, a short range coupling, a local conservation rule, and a slow driving by an external perturbation. The dynamics of these models can be regarded as a discrete version of a driven diffusion equation of a potential $E(\boldsymbol{x},t)$

$$\frac{\partial E}{\partial t}+\nabla\cdot\boldsymbol{j}(E(\boldsymbol{x}))=\eta(\boldsymbol{x},t) \tag{1}$$

where $\boldsymbol{j}(E(\boldsymbol{x}))$ is the transport current and $\eta(\boldsymbol{x},t)$ is a stochastic driving term [10]. $\boldsymbol{j}(E(\boldsymbol{x}))$is a nonlinear function due to the local threshold dynamics. In order to understand the dynamical behavior of these systems analogies have been drawn to the theory of critical phenomena. Statistical techniques such as scaling theory or renormalization group methods are used to compute critical exponents and to explore universal properties [3,11,12].

A different approach to characterize the self-organized critical state is provided by the concepts of nonlinear dynamics. With these concepts it is possible to analyze the state space structure of the systems and to get a more detailed knowledge of the nature of the critical state as an attractor of the dynamics. Investigations of deterministic sandpile models [13,14], of effective Lyapunov exponents in an earthquake model [7], and of the spatiotemporal dynamics of Bloch walls [15] are examples in which such concepts are used.

In this paper we introduce a coupled map lattice model which is a generalization of the critical height sandpile automaton model (CHM) [4] and investigate its dependence upon certain system parameters and boundary conditions (BC's). For open BC's the system shows the properties we expect for the universality class of the CHM [11]. For periodic BC's there is a strong numerical evidence for a second order phase transition which takes place between states of different spatial order. We examine the dynamics and the spatial structure of the two phases. Analyzing the dynamical structure in the state space of the system we find a correlation between the phase transitition and the size of basins of attraction of some periodic attractors which are dominating the dynamics in the ordered phase. At the critical point transient lengths diverge and the transient dynamics turns out to be chaotic.

Our model is discrete in space and time but has an associated field quantity $E(\boldsymbol{x},t)$ which is continuous and to which we refer as a potential. It represents physical quantities such as the height of a sandpile, mechanical stress, etc. For simplicity of the notation of lattice sites we introduce a one-dimensional variable i which can be yielded through a mapping of the d-dimensional vector $\boldsymbol{x}$ onto one dimension. We represent the dynamics of the system by the equation

$$E_i(t+1)=E_i(t)+\frac{1}{2}\sum_{j=1}^{N} J_{ij}E_j(t)(\tanh\{g[E_j(t)-E_{\text{th}}]\}+1) + \eta(i,t), \tag{2}$$

where N is the number of cells, E_{th} is a threshold parameter, J_{ij} is the coupling matrice, $\eta(i,t)$ is a stochastic driving term, and g is a nonlinearity parameter. The explicit form of the tanh function is chosen to approximate the Heaviside step function which is used in the sandpile automaton models in the limit of $g\rightarrow\infty$. The elements of the coupling matrice J_{ij} are real and constant. J_{ij} describes the redistribution of E_j to site i when $E_j \gtrsim E_{\text{th}}$.

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We focus on the dependence of the model upon the local threshold E_{th}. The influence of the other parameters will be reported in a forthcoming paper [16]. Our simulations of Eq. (2) are carried out on a two-dimensional lattice of size $N=L\times L$. We consider only an isotropic next neighbor coupling and assume that the redistribution process is conservative, i.e., $\Sigma_{i=1}^{N} J_{ij}=0$, apart from the border of the system where conservation can be violated depending on BC's. Thus the elements of the coupling matrix are set to

$$J_{ii}=-1,$$

$$J_{j'i}=\frac{1}{4} \quad (j' \in \text{next neighbors}).$$

The nonlinearity parameter is held constant at $g=1000$. With this value the behavior of the system is expected to be very close to that in the limit $g\rightarrow\infty$. For open BC's Pietronero *et al.* have shown that in this asymptotic case the system belongs to the universality class of the CHM [11]. To confirm this we have carried out simulations of Eq. (2) with open BC's. The system is driven by a stochastic perturbation acting whenever the system has reached a stable configuration. Thus no interactions between different avalanches take place and the dynamics of an avalanche is governed only by the deterministic part of Eq. (2). After suffiently long transients the system moves into a stationary state with power law distributions of the sizes and the durations of avalanches. The critical exponents τ and δ for the distribution of sizes $D(s)\sim s^{-\tau}$ and the distribution of durations $D(t)\sim t^{-\delta}$ are $\tau=1.25\pm0.03$ and $\delta=1.28\pm0.02$ where the size s of an avalanche with duration t is defined as

$$s=\sum_{t'=0}^{t}\sum_{i=1}^{N}[E_i(t')-E_i(t'-1)]^2.$$

Indeed these results are consistent with the calculations in [17] and the numerical estimates in [4]. Furthermore, we have calculated the critical ratio $\rho=\langle E\rangle/E_{th}$ of the average potential per site $\langle E\rangle$ and the threshold E_{th}. The theoretical value of $\rho\approx0.62$ [17] is universal for the class of the CHM due to the invariance of Eq. (2) under the transformation $E_i\rightarrow E_i'=E_i/E_{th}$ [18]. For $L\rightarrow\infty$ we find a ratio $\rho=0.63\pm0.02$ which agrees well with the theoretical value.

For periodic BC's we find a different behavior of the system. First the driving mechanism should be reconsidered because now the system cannot dissipate energy. If the same driving mechanism is used as for open BC's, the total energy of the system E_{tot} will tend to increase, i.e., the system becomes nonstationary. But since the dynamics of avalanches in the open system is not affected by the infinitely slow driving, we are only interested in the dynamics of the relaxation process. Thus we investigate the periodic system without driving. Then the total energy

$$E_{tot}=\sum_{i=1}^{N}E_i(t)=N\langle E\rangle=\text{const}$$

is a constant of motion. Therefore the dynamics takes place on a finite $(N-1)$-dimensional hyperplane in state space

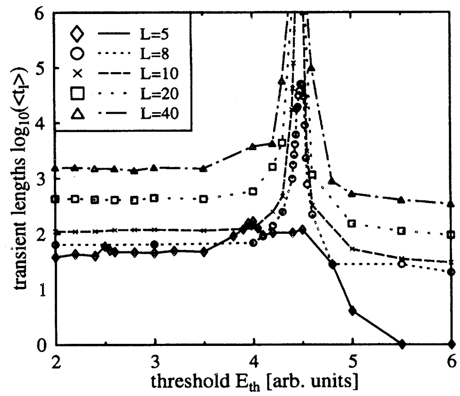

FIG. 1. Logarithmic plot of the average of transient lengths t_l computed on the basis of 10 000 runs for each value of E_{th}.

which is spanned by N basis vectors corresponding to sites of the lattice. It is finite because the value of E at each site is limited to the interval $[0,E_{tot}]$.

Without loss of generality we set $\langle E\rangle=3$. We analyze the transients and the final states of the system, starting from randomly chosen initial states on the hyperplane defined by $\langle E\rangle$. For all E_{th} we find a large number of coexisting attractors. We compute an entropy H which we define as

$$H(E_{th},L)=-\sum_{i=1}^{\nu}p_{S_i}\ln p_{S_i}+\sum_{i=1}^{\nu}p_{S_i}\ln T_i-\sum_{i=1}^{\nu}\frac{p_{S_i}}{T_i}\sum_{t=1}^{T_i}\int p_E(S_i,t)\ln p_E(S_i,t)dE, \quad (3)$$

where $\nu=\nu(L)$ is the number of attractors in the system [19]. $p_{S_i}(E_{th})$ is the probability of finding an attractor S_i with period T_i and $p_E(S_i,t)$ is the probability of a site to be in the state E of an attractor S_i at time t. The first term of Eq. (3) represents the complexity due to coexisting attractors, the second term stems from the periodicity of the attractors, and the third term represents the entropy of the spatial configuration averaged over one period. For varying thresholds E_{th} and system sizes L we find that the function is continuous and that at $E_{th}=E_{th}^{*}\approx4.5$ the slope of $H(E_{th},L)$ becomes very large. We interpret this result as an indication of a second order phase transition.

For $E_{th}<E_{th}^{*}$ the attractors are coexisting stable fixed points and periodic states. The related spatial structures of these attractors are regular patterns, e.g., the most dominant attractor to which we refer as S_1 shows a checkered pattern alternating between $E=0$ and $E=2\langle E\rangle$. Thus the subcritical regime can be identified with the ordered phase. With growing parameter E_{th} new attractors appear and previously stable ones become unstable. The temporal and spatial behavior of the attractors becomes more complex but still is regular, e.g., there are standing waves, running waves, or running pulses. Near the critical point the most dominant attractor is a single plane running wave moving on the checkered pattern of S_1 as background. We call this attractor S_2.

Figure 1 shows the average transient lengths computed for

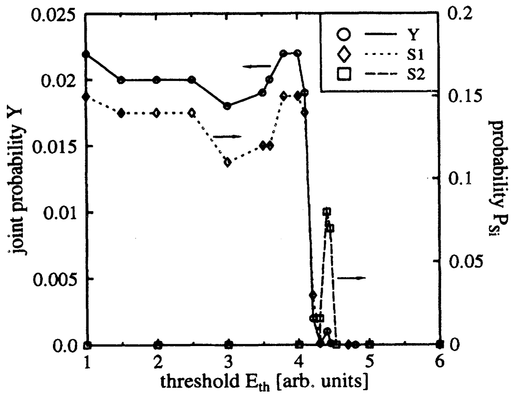

FIG. 2. Joint probability distribution Y and probabilities P of the two most dominant attractors S_1 and S_2 which account for the details in the structure of Y.

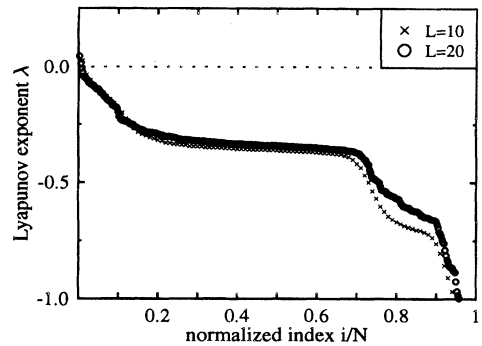

FIG. 3. Normalized Lyapunov spectra for two system sizes with $N=L\times L$.

uniform randomly chosen initial conditions. At $E_{th}=E^*_{th}$ the average transient lengths as well as the variance diverge, so that most of the final states cannot be seen even for relatively small lattice sizes. To understand this we examine the probability Y that two random initial conditions end upon the same attractor [20]. Y is defined by $Y=\Sigma_i(p_{S_i})^2$. If $Y\rightarrow 0$ as $L\rightarrow\infty$ this means that all basins of attraction become infinitely small. On the contrary, if Y remains finite there are a few big basins which fill almost the whole of state space. In Fig. 2 we have plotted Y extrapolated for $1/L\rightarrow 0$. It can be seen that Y vanishes at the critical point. We have also plotted the probabilities p for the attractors S_1 and S_2. The two distributions account for the details in the structure of Y. These results show that the basins of attraction vanish at the critical point although a stability analysis shows that there still are stable attractors.

Because the attractor S_1 is dominating the dynamics in the subcritical regime we define the order parameter η in analogy to antiferromagnets as

$$\eta(E_{th},L)=\sum_{i=1}^{\nu}\frac{p_{S_i}}{T_i}\sum_{t=1}^{T_i}\left|\sum_{j=1}^{N}f(i)E_j\right|, \tag{4}$$

where $f(i)$ defines the sign of the summands which corresponds to a two-dimensional checkered pattern. Indeed η vanishes near the critical point and is zero for $E_{th}>E^*_{th}$. Since transient lengths diverge near E^*_{th}, however, the scaling of η could not be determined numerically. Nevertheless we regard these results as strong numerical evidence of a second order phase transition. Carlson and co-workers found that in the unpertubated, periodic case self-organized critical systems obey a diffusion equation which has a singularity in the diffusion coefficient [12]. The driven, open system converges to this critical point as the size L diverges. In our model with periodic BC's the critical ratio ρ has a value of $\rho=0.67\pm0.01$, independent of $\langle E\rangle$. This ratio agrees remarkably well with that for open BC's. Therefore we regard the observed phase transition as the source of the diffusion singularity in our model.

Above the critical point transients become shorter again (see Fig. 1). In most of the final states there are fixed points with E at all sites below the threshold E_{th}. Their spatial structures are strongly irregular, hence the supercritical regime is the disordered phase of the system. We have analyzed the equal-time spatial correlation function $G(r)=\langle(E_i-\langle E\rangle)(E_j-\langle E\rangle)\rangle$ of the fixed points for varying system sizes L and thresholds E_{th}. $r=L^{-1}|x_i-x_j|$ is the normalized distance of cells i and j on the two-dimensional lattice. The normalized correlation function $G'(r)=G(r)/G(r=0)$ is independent of L and decays linearly. We have also computed spatial correlations with the energies E_i randomly redistributed on the lattice. In this case no correlations have been found. This result can be interpreted as a hint that the spatial structure in the supercritical regime is governed by a deterministic rule similar to the case reported in [15].

Because transients become very long near the critical point we have studied the dynamics of transient states. At the critical point we find that the largest Lyapunov exponents are positive with values of $(2.5\pm0.5)\times10^{-2}$ and $(1.7\pm0.5)\times10^{-2}$. The Lyapunov exponents are calculated by means of QR decomposition [21], i.e., the decomposition of the Jacobian matrix of the deterministic part in Eq. (2) into an orthogonal matrix Q and an upper triangular matrix R, and singular value decomposition [23], respectively. The Lyapunov spectra [22] plotted for an index normalized by N are shown in Fig. 3. They are invariant at least up to system sizes of $L=20$. This indicates that transient dynamics is governed by a chaotic repeller [25].

As long as avalanches do not reach the boundary, the dynamics of avalanches in the open system also takes place on a plane of constant energy in state space. This plane is defined by the actual total energy of the system. We therefore expect that the invariant sets in state space which dominate the relaxation dynamics in the system with periodic BC's are also relevant for the avalanche dynamics. To prove this we compare the short time dynamics of avalanches in the open system with that of transients in the periodic case. The analysis is done by means of effective Lyapunov exponents $\lambda_{eff}(t)$ which are a measure for the divergence rate of initially adjacent trajectories for finite times t. Their distribution in state space is related to the stable manifolds of the invariant sets [24]. Figure 4 shows the distributions of the largest λ_{eff} for different times t and different BC's. For both

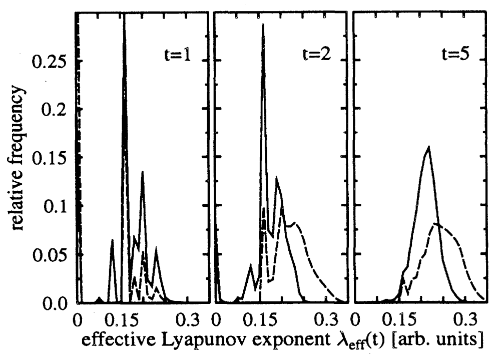

FIG. 4. Frequency distribution of effective Lyapunov exponents $\lambda_{eff}(t)$ for three different times in the open (full line) and in the periodic case (broken line). For open BC's $\lambda_{eff}(t)$ has been computed for points in state space belonging to avalanches, whereas for periodic BC's only transient states have been considered.

systems the size is $L=10$ and the threshold E_{th} is set to 4.5. For the periodic system $\langle E \rangle$ is set to 2.55 which is equal to the average energy per site in the stationary state of the corresponding open system. It is clearly visible that for short times both distributions are highly correlated. For longer times there are deviations which are likely due to the influence of the BC's.

In summary, we have shown that a coupled map lattice model belonging to the universality class of the CHM reveals a second order phase transition in dependence on the local threshold and under the condition of periodic BC's. At the critical point dynamics is dominated by a chaotic repeller and transient lengths diverge. For open BC's the system self-organizes to the vicinity of this critical point. The analysis of the dynamics of avalanches on short time scales suggests that the avalanches can be understood as chaotic transitions between metastable states. Therefore we believe that the occurrence of SOC, at least in this model, is related to the structure of invariant sets in state space. Basically this phenomenon may be similar to the occurrence of $1/f$ noise in the case of anomalous diffusion [26] where scaling is caused by the fractal properties of a chaotic transport process.

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